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INTERNAL GRAVITY WAVES IN CRITICAL GENERATION MODES AND IN THE VICINITY OF TRAJECTORIES OF MOTION OF PERTURBATION SOURCES

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The field of internal gravity waves in a layer of an arbitrary stratified fluid is studied for critical generation modes and in the vicinity of trajectories of motion of the perturbation sources. The exact solutions describing the structure of a separate mode of the wave field in the vicinity of the perturbation source in the critical generation modes are investigated, and expressions for the total field representing the sum of all wave modes are obtained. In the vicinity of the trajectories of the perturbation sources, asymptotic representations of the eigenfunctions and eigenvalues of the basic vertical spectral problem of internal waves are constructed in the approximation of large wave numbers and asymptotic expressions for a separate mode of the wave field are obtained that describe the spatial structure and features of the fields of internal gravity waves.

Key words: *internal gravity waves, stratified medium, asymptotics of a separate mode, group velocity.*

Introduction. The recent increased interest in the dynamics of wave motion of various inhomogeneous, in particular, stratified, fluids [1–7] has been motivated by the need to solve problems of geophysics, oceanology, atmosphere physics, environmental protection and research, use of cryogenic fluids in engineering, operation of complex hydraulic engineering structures, including sea oil-extracting complexes, and some other urgent problems of science and engineering. As a rule, these problems have been investigated by means of asymptotic methods. Unperturbed equations of hydrodynamics are used to derive asymptotic expansions or anzatzes (in German, Anzatz is a kind of solution), which are then employed to solve problems for perturbed equations that can take into account nonlinear, nonuniform, and nonstationary effects of natural stratified media.

One of the mechanisms whereby internal gravity waves can be generated in the ocean is by generating wave fields during motion (flow) of solids, spots of turbulence, water lenses, and other nonwave formations with abnormal characteristics and other nonlocal perturbation sources of wave fields [1–3]. As a rule, this formulation is used for the linear problem of the steady-state field of internal gravity waves generated by motion of a local perturbation source in a stratified fluid layer of thickness H with an arbitrary depth distribution of the Brunt–Väisälä frequency $N^2(z) = -gd\ln\rho/dz$ (g is the acceleration due to gravity; ρ is the density of the stratified medium). The thus obtained solutions in the form of multiple quadratures even within the framework of linear models are original and determine nontrivial physical consequences [1, 2, 8–10].

It should be noted that, for a detailed description of a wide range of physical phenomena due to the dynamics of stratified horizontally nonuniform and nonstationary media, it is necessary to use advanced mathematical models, which, as a rule, are rather complex, nonlinear, and multiparameter and can be investigated completely only using numerical methods. In some cases, however, an initial qualitative understanding of the phenomena in question can be obtained on the basis of simpler asymptotic models and analytical methods [1–3]. Therefore, too study

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all wave effects, it is sufficient, as a rule, to construct relatively simple models that can be studied theoretically. Subsequently, these models are included in a set of blocks forming the general picture of wave dynamics that allows a study of the relationship between various wave phenomena. In some cases, however, despite the simplicity of the modeling assumptions used, a successful choice of the form of solution provides physically substantial results. In this connection, mention should be made of the problem of the evolution of nonharmonic wave trains in a horizontally smoothly-nonuniform and a nonstationary stratified medium [1, 2]. The modeling solutions constructed to describe the structure of the fields near the wave fronts of separate modes in a vertically stratified medium allow one to obtain asymptotic representations of internal-wave fields taking into account the variability of the medium not only along the vertical and horizontal but also in time. In addition, such solutions are in good agreement with results of full-scale observations of wave fields [11].

From the aforesaid, it is of interest to study the previously unstudied exact solutions describing the near field of internal gravity waves for critical generation modes and asymptotic solutions for far wave fields in the vicinity of trajectories of motion of perturbation sources.

1. Near Field of Critical Generation Modes. We consider an elevation η of internal gravity waves generated by a point source of mass of unit intensity, which begins to move at the time $t = 0$ in a stratified-fluid layer ($-H < z < 0$). This elevation is determined from the problem [1, 2]

$$L\eta = \theta(t)\,\frac{\partial^2}{\partial t\,\partial z_0}\,(\delta(x - x_0(t))\delta(y - y_0(t))\delta(z - z_0(t)), \tag{1.1}$$

where

$$L = \frac{\partial^2}{\partial t^2}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big) + N^2(z)\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big),$$

$N(z)$ is the Brunt–Väisälä frequency, $\theta(t) = 0$ at $t < 0$, $\theta(t) = 1$ at $t > 0$, and $[x_0(t), y_0(t), z_0(t)]$ is the trajectory of motion of the source. As boundary conditions we use the rigid cover approximation:

$$\eta = 0, \qquad z = 0, -H. \tag{1.2}$$

Solution (1.1), (1.2) has the form [1, 2]

$$\eta = \sum_{n=1}^{\infty} \eta_n,$$

$$\eta_n = \frac{1}{2\pi}\,\mathrm{Re}\int\limits_0^{\infty} \frac{\omega_n^2(k)}{k}\,\varphi_n(z,k)\int\limits_0^{t} \frac{\partial\varphi_n(z_0(\tau),k)}{\partial z_0(\tau)}\,\exp(i\omega_n(k)(t-\tau))J_0(kr(\tau))\,d\tau\,dk,$$

$$r(\tau) = [(x - x_0(\tau))^2 + (y - y_0(\tau))^2]^{1/2},$$

where J_0 is a zero-order Bessel function and $\omega_n(k)$ and $\varphi_n(z,k)$ are the eigennumbers and eigenfunctions of the basic spectral vertical problem of internal waves:

$$\frac{\partial^2\varphi_n(z,k)}{\partial z^2} + k^2\Big(\frac{N^2(z)}{\omega_n^2(k)} - 1\Big)\varphi_n(z,k) = 0, \qquad \varphi_n = 0, \quad z = 0, -H.$$

In the case of a steady-state mode and rectilinear uniform motion of the point perturbation source at velocity V at constant depth ($z_0 = \mathrm{const}$, $y_0 = 0$, and $x_0 = -V\tau$), a separate mode of the elevation η_n in a coordinate system moving together with the source ($x + Vt \equiv \lambda$) have the form [1, 2]

$$\eta_n(\lambda,y) = \frac{1}{2\pi V}\int\limits_0^{\infty} \frac{\omega_n^2(k)}{k}\,\varphi_n(z,k)\,\frac{\partial\varphi_n(z_0,k)}{\partial z_0}\int\limits_{-\infty}^{\lambda} \cos\Big(\frac{\omega_n(k)(\lambda-\xi)}{V}\Big)J_0\Big(k\sqrt{y^2+\xi^2}\Big)\,d\xi\,dk. \tag{1.3}$$

For small values of λ and y, i.e., in the vicinity of the moving perturbation source, the separate elevation mode $\eta_n(\lambda,y)$ can be represented as

$$\eta_n(\lambda,y) = \eta_n(0,0) + T_n\lambda + B_n y + \ldots\ ,$$

$$T_n = \frac{\partial \eta_n}{\partial \lambda}(0,0), \qquad B_n = \frac{\partial \eta_n}{\partial y}(0,0). \tag{1.4}$$

Obviously, because of the symmetry of the problem in the variable y, the function $\eta_n(\lambda, y)$ is even for this variable and, hence, $B_n = 0$.

We examine the behavior of the function $\eta_n(0,0)$. The internal integral in (1.3) is taken in the form [12, 13]

$$R_n \equiv \int_0^\infty \cos\Big(\frac{\omega_n(k)\xi}{V}\Big) J_0(k\xi)\, d\xi = \frac{1}{\mu_n^+(k)}, \quad k > \frac{\omega_n(k)}{V}, \qquad R_n = 0, \quad k < \frac{\omega_n(k)}{V},$$

$$\mu_n^\pm(k) = \sqrt{\pm k^2 \mp \omega_n^2(k)/V^2}.$$

Then,

$$\eta_n(0,0) = \int_{d_n}^\infty \frac{\omega_n^2(k)}{k\mu_n^+(k)}\, F_n(k,z,z_0)\, dk,$$

where

$$F_n(k,z,z_0) = \frac{1}{2\pi V}\, \varphi_n(z,k)\, \frac{\partial \varphi_n(z_0,k)}{\partial z_0},$$

d_n is a root of the equation $k^2V^2 = \omega_n^2(k)$ for $V < c_n$ and $d_n = 0$ for $V > c_n$; $c_n = d\omega_n(k)/dk$ $(k = 0)$ is the maximum group velocity of the nth mode.

We consider the case of an exponentially stratified fluid [$N(z) = \text{const}$]:

$$F_n(k,z,z_0) = \frac{\pi}{VN^2H^2} \sin\Big(\frac{\pi n z}{H}\Big) \cos\Big(\frac{\pi n z_0}{H}\Big) \equiv A_n(z,z_0).$$

Let $\varepsilon_n^\pm \equiv b_n\sqrt{\pm 1 \mp c_n^2/V^2}$ be a measure of the deviation of the source velocity V from the quantities $c_n = NH/(\pi n)$ and $b_n = \pi n/H$.

In the case of a supercritical mode $(V > c_n)$, the function $\eta_n(0,0)$ can be expressed as

$$\eta_n(0,0) = A_n(z,z_0)\int_0^\infty \frac{dk}{\sqrt{k^2+b_n^2}\,\sqrt{k^2+(\varepsilon_n^+)^2}} = \frac{A_n(z,z_0)}{b_n}\, K\Big(\frac{\sqrt{b_n^2-(\varepsilon_n^+)^2}}{b_n}\Big)$$

$$= \frac{1}{\pi N^2 HV}\, K\Big(\frac{c_n}{V}\Big) \sin\Big(\frac{\pi n z}{H}\Big) \cos\Big(\frac{\pi n z_0}{H}\Big),$$

and in the case of a subcritical mode $(V < c_n)$,

$$\eta_n(0,0) = A_n(z,z_0)\int_{\varepsilon_n}^\infty \frac{dk}{\sqrt{k^2+b_n^2}\,\sqrt{k^2-(\varepsilon_n^-)^2}} = \frac{A_n(z,z_0)}{\sqrt{b_n^2+(\varepsilon_n^-)^2}}\, K\Big(\frac{b_n}{\sqrt{b_n^2+(\varepsilon_n^-)^2}}\Big)$$

$$= \frac{1}{\pi N^2 H c_n}\, K\Big(\frac{V}{c_n}\Big) \sin\Big(\frac{\pi n z}{H}\Big) \cos\Big(\frac{\pi n z_0}{H}\Big).$$

Here $K(x)$ is the total elliptic integral of the first kind [12, 13]:

$$K(x) = \int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - x^2\sin^2\tau}}.$$

We calculate the coefficient T_n in (1.4), which by virtue of the definition, to within V, is a value of the separate mode W_n of the vertical velocity at $\lambda = y = 0$:

$$\frac{\partial \eta_n}{\partial \lambda} = \frac{1}{V}\frac{\partial \eta_n}{\partial t} = \frac{1}{V} W_n, \qquad T_n = \frac{W_n(0,0)}{V},$$

$$\frac{\partial \eta_n(0,0)}{\partial \lambda} = \frac{1}{2\pi V} \int_0^\infty \frac{\omega_n^2(k)}{k} \varphi_n(z,k) \frac{\partial \varphi_n(z_0,k)}{\partial z_0} J_0\Big(k\sqrt{y^2+\lambda^2}\Big)\, dk$$

$$-\frac{1}{2\pi V} \int_0^\infty \frac{\omega_n^2(k)}{k} \varphi_n(z,k) \frac{\partial \varphi_n(z_0,k)}{\partial z_0} \int_{-\infty}^{\lambda} \frac{\omega_n(k)}{V} \sin\Big(\frac{\omega_n(k)(\lambda-\xi)}{V}\Big) J_0\Big(k\sqrt{y^2+\xi^2}\Big)\, d\xi\, dk \equiv P_{1n} - P_{2n}.$$

The term P_{1n} is represented as

$$P_{1n}(\lambda, y) = \frac{1}{2\pi V} \int_0^\infty \frac{kN^2}{k^2+b_n^2} \varphi_n(z,k) \frac{\partial \varphi_n(z_0,k)}{\partial z_0} J_0\Big(k\sqrt{y^2+\lambda^2}\Big)\, dk$$

$$= \frac{n}{H^2 V} \sin\Big(\frac{\pi n z}{H}\Big) \cos\Big(\frac{\pi n z_0}{H}\Big) K_0\Big(\frac{\pi n}{H}\sqrt{y^2+\lambda^2}\Big),$$

where $K_0(x)$ is a zero-order McDonald function [12, 13].

Summing the series $\sum\limits_{n=1}^{\infty} P_{1n}$, we obtain the expression

$$P_1(\lambda, y) \equiv \sum_{n=1}^{\infty} P_{1n}(\lambda, y) = \frac{1}{4\pi V}\Big[\frac{z_-}{(r^2+z_-^2)^{3/2}} + \frac{z_+}{(r^2+z_+^2)^{3/2}}$$

$$- \sum_{m=1}^{\infty} \Big(\frac{2mH - z_-}{(r^2+(2mH-z_-)^2)^{3/2}} - \frac{2mH + z_-}{(r^2+(2mH+z_-)^2)^{3/2}}$$

$$+ \frac{2mH - z_+}{(r^2+(2mH-z_+)^2)^{3/2}} - \frac{2mH + z_+}{(r^2+(2mH+z_+)^2)^{3/2}}\Big)\Big]$$

($r = \sqrt{\lambda^2+y^2}$, $z_- = z - z_0$, and $z_+ = z + z_0$) which can be interpreted as follows. The resultant field is the sum of the fields of the rereflected sources (with respect to the boundaries $z = mH$, $m = 0, \pm 1, \pm 2, \ldots$) , which are at the points $z_m^{\pm} = \pm(z_0 + 2mH)$, $m = 0, \pm 1, \pm 2, \ldots$, and the field of each source is expressed in terms of the derivative of the fundamental solution of the three-dimensional Laplace equation with respect to z_0 in free space.

For $r = 0$, the series $\sum\limits_{n=1}^{\infty} P_{1n}$ is also summed up and expressed in terms of the derivative $\Psi'(x)$:

$$P_1(0,0) = \frac{1}{16\pi H^2 V}\Big(\Psi'\Big(-\frac{z_-}{2H}\Big) + \Psi'\Big(-\frac{z_+}{2H}\Big) - \Psi'\Big(\frac{z_-}{2H}\Big) - \Psi'\Big(\frac{z_+}{2H}\Big) + \frac{4H^2}{z_-^2} + \frac{4H^2}{z_+^2}\Big);$$

$\Psi(x) = \ln\Gamma(x)'$ is the psi function or the derivative of the gamma function [12, 13].

We now consider the term $P_{2n}(0,0)$:

$$P_{2n}(0,0) = \frac{nN}{H^2V^2} \sin\Big(\frac{\pi n z}{H}\Big) \cos\Big(\frac{\pi n z_0}{H}\Big) \int_0^\infty \frac{k^2}{(k^2+b_n^2)^{3/2}} \int_0^\infty \sin\Big(\frac{\omega_n(k)\xi}{V}\Big) J_0(k\xi)\, dk\, d\xi. \tag{1.5}$$

The internal integral in (1.5) is taken in the form

$$Q_n \equiv \int_0^\infty \sin\Big(\frac{\omega_n(k)\xi}{V}\Big) J_0(k\xi)\, d\xi = \frac{1}{\mu_n^-(k)}, \quad \frac{\omega_n(k)}{V} > k, \qquad Q_n = 0, \quad \frac{\omega_n(k)}{V} < k.$$

We next obtain

$$P_{2n}(0,0) = \frac{nN}{H^2V^2} \sin\Big(\frac{\pi n z}{H}\Big) \cos\Big(\frac{\pi n z_0}{H}\Big) \int_0^{\varepsilon_n^-} \frac{k\, dk}{(k^2+b_n^2)\sqrt{(\varepsilon_n^-)^2 - k^2}}.$$

As a result, we have

$$P_{2n}(0,0) = \frac{nN}{H^2V^2} \sin\Big(\frac{\pi n z}{H}\Big) \cos\Big(\frac{\pi n z_0}{H}\Big) \ln\Big(\frac{c_n}{V} + \sqrt{\frac{c_n^2}{V^2} - 1}\Big).$$

It should be noted that, because of the properties of the integrals Q_n and a decrease in the values of the maximum group velocities $c_1 > c_2 > c_3 > \ldots$ with decreasing mode number, the series $\sum\limits_{n=1}^{\infty} P_{2n}(0,0)$ has a finite number of nonzero terms.

Thus, the asymptotic and exact representations obtained for the separate mode and the total field allow one to describe the critical modes of generation of internal gravity waves near perturbation sources moving at arbitrary velocities.

2. Wave Fields near the Trajectories of Motion of Perturbation Sources. For convenience and because of linearity of problem (1.1), we study a solution G of this problem ($\partial G/\partial z_0 = \eta$) that, at $t \to \infty$ and fixed $\xi = x + Vt$ has the form [1, 2]

$$G(\xi, y, z, z_0) = \sum_n G_n = \sum_n (J_n^+ + J_n^-),$$

where

$$J_n^{\pm} = \operatorname{Re} \frac{V}{2\pi} \int\limits_0^{\infty} \frac{\omega_n^2(k)\varphi_n(z,k)\varphi_n(z_0,k) \exp\,(i(\pm\mu_n(\nu)\xi - \nu y))}{k^2(V^2 - \omega_n(k)\omega_n'(k)/k)}\, d\nu,$$
$$k(\nu) = \sqrt{\mu_n^2(\nu) + \nu^2}, \qquad \mu_n(k(\nu)) = \omega_n(k(\nu))/V. \tag{2.1}$$

In (2.1), it is reasonable to pass to the integration variable k:

$$J_n^{\pm} = \operatorname{Re} \frac{1}{2\pi V} \int\limits_0^{\infty} \frac{\omega_n^2(k)\varphi_n(z,k)\varphi_n(z_0,k) \exp\,(i(\pm\omega_n(k)\xi/V - \nu_n(k)y))}{k\nu_n(k)}\, dk, \tag{2.2}$$

$$\nu_n(k) = \sqrt{k^2 - \omega_n^2(k)/V^2}.$$

We examine the field of internal gravity waves for $\xi \to \infty$ and for bounded (or arbitrarily small) y, i.e., in the vicinity of the trajectories of motion of the perturbation sources. In this case, to study the behavior of the solution $G(\xi, y, z, z_0)$, it is necessary to take into account the integrals J_n^+ and J_n^- and the fact that the conventional method of stationary phase is unsuitable for calculating the wave field since the stationary points in (2.2) tend to infinity. In addition, as $k \to \infty$, the functions $\varphi_n(z,k)$ cannot be treated as slowly varying functions of the variable k; for large k, these functions are concentrated in the vicinity of the value z^* providing a maximum for $N(z)$, and with distance from the value z^*, the functions $\varphi_n(z,k)$ decrease rapidly [1, 8]. These properties of the eigenfunctions allow Eq. (2.2) to be reduced to known reference integrals.

Because for $k \to \infty$, the functions $\varphi_n(z,k)$ are concentrated in the vicinity of the maximum of $N(z)$ and decrease rapidly outside this vicinity, for large k, the real distribution $N(z)$ with a distinct thermocline can be replaced by a model distribution with a quadratic function $N(z)$, and the boundary condition at $z = 0, H$ can be replaced by the condition of exponential decrease of $\varphi_n(z,k)$ with distance from the maximum point, i.e., $\varphi_n(+\infty,k) = \varphi_n(-\infty,k) = 0$.

Below, we consider a model distribution of the buoyancy frequency $N^2(z)$ that approximates the stratification with one thermocline maximum typical of the ocean [1–3]:

$$N^2(z) = N_0^2 - 4\chi^2(z - z^*)^2. \tag{2.3}$$

Here $z = z^*$ is the maximum point of $N^2(z)$. To simplify the transformations, we set $z^* = 0$.

To find the eigenfunctions $\varphi_n(z,k)$ and the dispersion curves $\omega_n(k)$ in the case of stratification (2.3), we write the corresponding boundary-value problem as

$$\frac{\partial^2\varphi_n(z,k)}{\partial z^2} + \frac{k^2}{\omega_n^2(k)}\,(N_0^2 - 4\chi^2 z^2 - \omega_n^2(k))\varphi_n(z,k) = 0,$$
$$\varphi_n(+\infty,k) = \varphi_n(-\infty,k) = 0. \tag{2.4}$$

Performing the substitution $z = \alpha_n x$ (the coefficient α_n is to be determined), we obtain

$$\varphi_n(z) = \varphi_n(\alpha_n x) = \psi(x).$$

Then, expression (2.4) becomes

$$\frac{\partial^2 \psi_n(x)}{\partial x^2} + \Big(\frac{\alpha_n^2 k^2 (N_0^2 - \omega_n^2(k))}{\omega_n^2(k)} - \frac{4\alpha_n^4 \chi^2 k^2}{\omega_n^2(k)}\, x^2\Big)\psi_n(x) = 0,$$
$$\psi_n(+\infty, k) = \psi_n(-\infty, k) = 0. \tag{2.5}$$

If the conditions

$$\frac{4\alpha_n^4 \chi^2 k^2}{\omega_n^2(k)} = 1, \qquad \frac{\alpha_n^2 k^2 (N_0^2 - \omega_n^2(k))}{\omega_n^2(k)} = 2n+1, \qquad n = 0, 1, 2, \dots\ , \tag{2.6}$$

are satisfied, Eq. (2.5) coincides with the equation

$$\frac{\partial^2 \psi_n(x)}{\partial x^2} + (\lambda - x^2)\psi_n(x) = 0, \qquad \lambda = 2n+1,$$

whose solution are Chebyshev–Hermite functions

$$\psi_n(x) = H_n(x)\, \exp{(-x^2/2)};$$

$H_n(x)$ is a Hermite polynomial of the nth degree [12, 13]. Therefore, solution (2.4) becomes

$$\varphi_n(z, k) = B_n H_n(z/\alpha_n) \exp{(-z^2/(2\alpha_n^2))}, \tag{2.7}$$

where according to (2.6),

$$\alpha_n(k) = \sqrt{\omega_n(k)/(2\chi k)}\,; \tag{2.8}$$

$$\omega_n(k) = \Big(\sqrt{\chi_n^2 + k^2 N_0^2} - \chi_n\Big)/k, \qquad \chi_n = \chi(2n+1). \tag{2.9}$$

From the normalization condition

$$B_n^2 = \int\limits_{-\infty}^{\infty} (N_0^2 - 4\chi^2 z^2) H_n^2\Big(\frac{z}{\alpha_n}\Big) \exp\Big(-\frac{z^2}{\alpha_n^2}\Big)\, dz = 1,$$

we find the constant B_n:

$$B_n = [\alpha_n \|H_n\|^2 (N_0^2 - 4\chi^2 \alpha_n^2 (n + 1/2))]^{-1/2}; \tag{2.10}$$

$\|H_n\| = \sqrt{2^n n! \sqrt{\pi}}$ is the norm of the Hermite polynomial of the nth degree [12, 13].

Thus, the orthonormalized eigenfunctions $\varphi_n(z, k)$ and the dispersion curves $\omega_n(k)$ are defined by formulas (2.7) and (2.9), respectively, and $\alpha_n(k)$ is determined from (2.8).

We calculate the asymptotics of the integrals $J_n^{\pm}$ as $\xi \to \infty$. We note that, for fixed y, z, and z_0, successive integration of the integrand in (2.2) by parts gives the estimate $O(\xi^{-\infty})$ since all substitutions outside the integral vanish (in the lower limit, by virtue of the evenness of the integrand amplitude in k and the oddness of the phase function, and in the upper limit, by virtue of the exponential decrease in the eigenfunction $\varphi_n(z, k)$) in k.

Let us consider asymptotics (2.2) for $\xi \to \infty$ and small z, z_0, and y. We note that the field of a distributed source (having finite dimensions) rather than of a point source has a physical meaning, i.e., the expression for the field given below require averaging over the source coordinates, in particular, along the vertical coordinate z_0. Since, with this averaging, the contributions of each oscillating mode for $y/\xi \to 0$ and $\xi \to \infty$ are negligibly small (because the integral on which they oscillate tends to zero at $k \to \infty$), we will further consider the mode $n = 0$, which plays the key role as $\xi \to \infty$ (the subscript n is omitted in this case).

The eigenfunction $\varphi(z, k)$ has the following form [see (2.7)–(2.10)]:

$$\varphi(z, k) = \Big(\frac{2\chi}{\pi}\Big)^{1/4} \Big(\frac{k}{\omega(k)}\Big)^{3/4} (k\omega(k) + \chi)^{-1/2} \exp\Big(-\frac{\chi k z^2}{\omega(k)}\Big). \tag{2.11}$$

Because the main contribution to the integral (2.2) is determined by large k, for $k > A$, where A is large enough, we expand the functions $\omega(k)$, $\nu(k)$, $\alpha^{-2}(k)$ in series

$$\omega(k) = N_0 - \chi k^{-1} + O(k^{-2}), \qquad \nu(k) = k - O(k^{-1}),$$
$$\alpha^{-2}(k) = 2\chi k/N_0 + 2\chi^2/N_0^2 + O(k^{-1}). \tag{2.12}$$

The integration region in (2.2) is divided into two regions: from 0 to A and from A to ∞. Then, for the integral in the first region, integration by parts gives the estimate $O(\xi^{-1})$. During integration in the second region, the functions in the integral (2.2) are replaced by their expansions (2.11) and (2.12). As a result, we have

$$J^{\pm} = \operatorname{Re} D \int\limits_A^{\infty} \exp\Big(-\frac{(z^2+z_0^2)\chi}{N_0}\, k - i\Big(\pm\frac{\chi\xi}{kV} + ky\Big)\Big)\frac{dk}{k^{3/2}},$$
$$D = \frac{\chi^{1/2}}{2^{1/2}\pi^{3/2} V N_0^{1/2}} \exp\Big(\pm i\xi\,\frac{N_0}{V}\Big). \tag{2.13}$$

In the integral (2.13), the lower limit is replaced by zero, but it is easy to see that this leads to an error of order $O(\xi^{-1})$. Next, making the change of variables $k = 1/u$, we obtain

$$J^{\pm} = \operatorname{Re} D \int\limits_0^{\infty} \exp\Big(-\frac{(z^2+z_0^2)\chi}{N_0 u} + i\Big(\mp\frac{\chi\xi}{V}\, u - \frac{y}{u}\Big)\Big)\frac{du}{u^{1/2}}. \tag{2.14}$$

Integrals (2.14) are expressed in elementary functions

$$\int\limits_0^{\infty} \frac{\exp\left(-pu - q/u\right)}{\sqrt{u}}\, du = \sqrt{\frac{\pi}{p}}\, \exp\Big(-2\sqrt{pq}\Big) \qquad (\operatorname{Re} p \geqslant 0, \quad \operatorname{Re} q \geqslant 0),$$

$$p = \pm i\chi\xi/V, \qquad q = (z^2+z_0^2)\chi/N_0 + iy,$$

where by the square root is meant the regular branch which takes positive values for positive arguments (the cut is located along the negative axis of the argument).

Using the notation

$$r = \sqrt{y^2 + [(z^2+z_0^2)\chi/N_0]^2}, \qquad \gamma = \sqrt{2\chi/V},$$

we write the asymptotics $G(\xi, y, z, z_0)$ for the higher mode ($n = 0$)) for $\xi \to \infty$ and small r as

$$G(\xi, y, z, z_0) = \frac{1}{\pi\sqrt{2VN_0\xi}}\Big[\exp\Big(-\gamma\sqrt{r+y}\sqrt{\xi}\Big)\cos\Big(\frac{\pi}{4} - \frac{N_0}{V}\,\xi - \gamma\sqrt{r-y}\sqrt{\xi}\Big)$$
$$+ \exp\Big(-\gamma\sqrt{r-y}\sqrt{\xi}\Big)\cos\Big(\frac{\pi}{4} - \frac{N_0}{V}\,\xi - \gamma\sqrt{r+y}\sqrt{\xi}\Big)\Big], \tag{2.15}$$

where the quantity $2\chi V^{-1} r\xi$ has order $O(1)$. Setting $y = 0$ in (2.15), we obtain

$$G = \frac{1}{\pi}\sqrt{\frac{2}{VN_0\xi}}\,\exp\Big(-\chi\sqrt{\frac{2(z^2+z_0^2)\xi}{N_0 V}}\Big)\cos\Big(\frac{\pi}{4} - \frac{N_0}{V}\,\xi - \chi\sqrt{\frac{2(z^2+z_0^2)\xi}{N_0V}}\Big).$$

If the source and the observation point are on the thermocline, i.e., $z = z_0 = 0$, we have

$$G = \frac{1}{\pi}\sqrt{\frac{2}{VN_0\xi}}\,\cos\Big(\frac{N_0}{V}\,\xi - \frac{\pi}{4}\Big).$$

From this, it follows that the wave field oscillating at frequency N_0/V decreases similarly to $\xi^{-1/2}$.

Thus, the obtained asymptotic solutions allow a description of the spatial structure of internal gravity waves near the trajectories of motion of perturbation sources. It is shown that, at large distances, the contribution of the higher generation modes to the wave field is negligibly small.

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